Depending on equations

A proof-relevant framework for unification in dependent type theory

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DistriNet – KU Leuven

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Unification for dependent types

Unification is used for many purposes:

logic programming, type inference, term rewriting, automated theorem proving, natural language processing, . . .

This talk:

checking definitions by dependent pattern matching

Disclaimer

My work is on dependently typed languages,
I know little about unification.

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$$(\texttt{suc}\ x = \texttt{zero}) \Rightarrow \bot$$

...but there will be types everywhere!

Dependent types: the 'big five'

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- Univalence: $(A \equiv B) \simeq (A \simeq B)$

and see how they interact with unification!

Depending on equations

Checking dependently typed programs

Unification in dependent type theory

Unification of dependently typed terms

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Why use dependent types?

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- ... guarantee that a program matches its specification
- ... use the same language for writing programs and proofs
- ... develop programs and proofs interactively

Dependent types

Per
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A **dependent type** is a family of types, depending on a term of a **base type**.

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A **dependent type** is a family of types, depending on a term of a **base type**.

e.g. `Vec` A n is the type of vectors of length n.

The Agda language

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All examples are (mostly) valid Agda code!

Using dependent types

With dependent types, we can give more precise types to our programs:

$$\texttt{replicate} : (n : \mathbb{N}) \to A \to \texttt{Vec}\ A\ n$$

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$$\texttt{replicate} : (n : \mathbb{N}) \to A \to \texttt{Vec}\ A\ n$$

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$$\texttt{append} : (m\ n : \mathbb{N}) \to \texttt{Vec}\ A\ m \to \texttt{Vec}\ A\ n \to \texttt{Vec}\ A\ (m + n)$$

Simple pattern matching

```
data ℕ : Type where
  zero : ℕ
  suc  : ℕ → ℕ
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minimum (suc x) zero    = { }
minimum (suc x) (suc y) = { }
```

Simple pattern matching

data $\mathbb{N}$: Type **where**
  zero : $\mathbb{N}$
  suc : $\mathbb{N} \to \mathbb{N}$

```
minimum : ℕ → ℕ → ℕ
minimum zero    y        = zero
minimum (suc x) zero     = zero
minimum (suc x) (suc y)  = { }
```

Simple pattern matching

$$\textbf{data}\ \mathbb{N} : \mathtt{Type}\ \textbf{where}$$
$$\quad \mathtt{zero} : \mathbb{N}$$
$$\quad \mathtt{suc} \ \ : \mathbb{N} \to \mathbb{N}$$

$$\mathtt{minimum} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$\mathtt{minimum}\ \mathtt{zero}\quad y \qquad = \mathtt{zero}$$
$$\mathtt{minimum}\ (\mathtt{suc}\ x)\ \mathtt{zero} = \mathtt{zero}$$
$$\mathtt{minimum}\ (\mathtt{suc}\ x)\ (\mathtt{suc}\ y) = \mathtt{suc}\ (\mathtt{minimum}\ x\ y)$$

Dependent pattern matching

$$\begin{array}{l}\textbf{data}\ \texttt{Vec}\ (A : \texttt{Type}) : \mathbb{N} \to \texttt{Type}\ \textbf{where} \\ \quad \texttt{nil}\ \ : \texttt{Vec}\ A\ \texttt{zero} \\ \quad \texttt{cons} : (n : \mathbb{N}) \to A \to \texttt{Vec}\ A\ n \to \texttt{Vec}\ A\ (\texttt{suc}\ n)\end{array}$$

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$$\texttt{tail} : (k : \mathbb{N}) \to \texttt{Vec}\ A\ (\texttt{suc}\ k) \to \texttt{Vec}\ A\ k$$
$$\texttt{tail}\ k\ \ xs \qquad\qquad = \{\ \}$$

Dependent pattern matching

```
data Vec (A : Type) : ℕ → Type where
  nil  : Vec A zero
  cons : (n : ℕ) → A → Vec A n → Vec A (suc n)

tail : (k : ℕ) → Vec A (suc k) → Vec A k
tail k  nil             = { } -- suc k = zero
tail k  (cons n x xs)   = { } -- suc k = suc n
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Dependent pattern matching

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data Vec (A : Type) : ℕ → Type where
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tail : (k : ℕ) → Vec A (suc k) → Vec A k
tail k  nil            = { }  -- impossible
tail k  (cons n x xs)  = { }  -- suc k = suc n
```

Dependent pattern matching

data Vec $(A : \texttt{Type}) : \mathbb{N} \to \texttt{Type}$ **where**
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tail : $(k : \mathbb{N}) \to \texttt{Vec}\ A\ (\texttt{suc}\ k) \to \texttt{Vec}\ A\ k$

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tail : $(k : \mathbb{N}) \to$ Vec A (suc k) $\to$ Vec A k

tail k (cons n x xs) = { } -- k = n

Dependent pattern matching

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$$\texttt{tail} : (k : \mathbb{N}) \to \texttt{Vec}\ A\ (\texttt{suc}\ k) \to \texttt{Vec}\ A\ k$$

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Specialization by unification

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The output of unification can change Agda's notion of equality!

Main question: How to make sure the output of unification is correct?

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Checking dependently typed programs

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Q: What is the fastest way to start a fight between type theorists?

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A: Mention the topic of equality.

The identity type

$$x \equiv_A y$$

. . . a dependent type depending on $x, y : A$.

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…type theory's built-in notion of equality.

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...type theory's built-in notion of equality.

...the type of **proofs** that $x = y$.

Operations on the identity type

$\texttt{refl} \quad : x \equiv_A x$

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$\texttt{sym} \quad : x \equiv_A y \to y \equiv_A x$

Operations on the identity type

$$\begin{aligned}
\texttt{refl} &: x \equiv_A x \\
\texttt{sym} &: x \equiv_A y \to y \equiv_A x \\
\texttt{trans} &: x \equiv_A y \to y \equiv_A z \to x \equiv_A z
\end{aligned}$$

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\texttt{trans} &: x \equiv_A y \to y \equiv_A z \to x \equiv_A z \\
\texttt{cong}\ f &: x \equiv_A y \to f\ x \equiv_B f\ y
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\texttt{sym} &: x \equiv_A y \to y \equiv_A x \\
\texttt{trans} &: x \equiv_A y \to y \equiv_A z \to x \equiv_A z \\
\texttt{cong}\ f &: x \equiv_A y \to f\ x \equiv_B f\ y \\
\texttt{subst}\ P &: x \equiv_A y \to P\ x \to P\ y
\end{aligned}$$

Unification problems as telescopes

A **unification problem** consists of

1. Flexible variables $x_1 : A_1$, $x_2 : A_2$, $\ldots$
2. Equations $u_1 = v_1 : B_1$, $\ldots$

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This can be represented as a **telescope**:

$$(x_1 : A_1)(x_2 : A_2)\ldots$$
$$(e_1 : u_1 \equiv_{B_1} v_1)(e_2 : u_2 \equiv_{B_2} v_2)\ldots$$

e.g. $(k : \mathbb{N})(n : \mathbb{N})(e : \texttt{suc}\ k \equiv_{\mathbb{N}} \texttt{suc}\ n)$

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$$\Gamma$$
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Unification problems as telescopes

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2. Equations $\bar{u} = \bar{v} : \Delta$

This can be represented as a **telescope**:

$$\Gamma(\bar{e} : \bar{u} \equiv_{\Delta} \bar{v})$$

e.g. $(k : \mathbb{N})(n : \mathbb{N})(e : \texttt{suc}\ k \equiv_{\mathbb{N}} \texttt{suc}\ n)$

Unifiers as telescope maps

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This can be represented as a *telescope map*:

$$f : \Gamma' \to \Gamma(\bar{e} : \bar{u} \equiv_{\Delta} \bar{v})$$

e.g. $f : () \to (n : \mathbb{N})(e : n \equiv_{\mathbb{N}} \texttt{zero})$
$f\ () = \texttt{zero}; \texttt{refl}$

Evidence of unification

A map $f : () \to (n : \mathbb{N})(e : n \equiv_{\mathbb{N}} \texttt{zero})$ gives us two things:

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$\Longrightarrow$ Unification is guaranteed to respect $\equiv$!

Three valid unifiers

$$f_1 : (k : \mathbb{N}) \to (k\ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n)$$
$$f_1\ k = k; k; \texttt{refl}$$

$$f_2 : () \to (k\ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n)$$
$$f_2\ () = \texttt{zero}; \texttt{zero}; \texttt{refl}$$

$$f_3 : (k\ n : \mathbb{N}) \to (k\ n : \mathbb{N})(e : k \equiv_{\mathbb{N}} n)$$
$$f_3\ k\ n = k; k; \texttt{refl}$$

Most general unifiers

A *most general unifier* of $\bar{u}$ and $\bar{v}$ is a unifier σ such that for any σ' with $\bar{u}\sigma' = \bar{v}\sigma'$, there is a ν such that $\sigma' = \sigma \circ \nu$.

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This is quite difficult to translate to type theory directly...

Intuition: if $f : \Gamma' \to \Gamma(\bar{e} : \bar{u} \equiv_\Delta \bar{v})$ is MGU, we can go back from $\Gamma(\bar{e} : \bar{u} \equiv_\Delta \bar{v})$ to Γ' without losing any information.

Equivalences

A function $f : A \to B$ is an **equivalence** if it has both a left and a right inverse:

$$\texttt{isLinv} : (x : A) \to g_1\ (f\ x) \equiv_A x$$
$$\texttt{isRinv} : (y : B) \to f\ (g_2\ y) \equiv_B y$$

In this case, we write $f : A \simeq B$.

Most general unifiers are equivalences!

$$f : \Gamma\left(\bar{e} : \bar{u} \equiv_{\Delta} \bar{v}\right) \simeq \Gamma'$$

Example of unification

$$(k\ n : \mathbb{N})(e : \texttt{suc}\ k \equiv_{\mathbb{N}} \texttt{suc}\ n)$$

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$$(k : \mathbb{N})$$

$$f : (k : \mathbb{N}) \to (k\ n : \mathbb{N})(e : \texttt{suc}\ k \equiv_{\mathbb{N}} \texttt{suc}\ n)$$
$$f\ k = k; k; \texttt{refl}$$

The solution rule

$$\texttt{solution} : (x : A)(e : x \equiv_A t) \simeq ()$$

The deletion rule

$$\texttt{deletion} : (e : t \equiv_A t) \simeq ()$$

The injectivity rule

$$\texttt{injectivity}_{\texttt{suc}} :$$
$$(e : \texttt{suc}\ x \equiv_{\mathbb{N}} \texttt{suc}\ y) \simeq (e' : x \equiv_{\mathbb{N}} y)$$

Negative unification rules

A *negative unification rule* applies to impossible equations, e.g. `suc` $x =$ `zero`.

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A *negative unification rule* applies to impossible equations, e.g. $\texttt{suc}\ x = \texttt{zero}$.

This can be represented by an equivalence:

$$(e : \texttt{suc}\ x \equiv_{\mathbb{N}} \texttt{zero}) \simeq \bot$$

where $\bot$ is the **empty type**.

The conflict rule

$$\texttt{conflict}_{\texttt{suc},\texttt{zero}} : (e : \texttt{suc}\ x \equiv_{\mathbb{N}} \texttt{zero}) \simeq \bot$$

The cycle rule

$$\mathtt{cycle}_{n,\mathtt{suc}\ n} : (e : n \equiv_{\mathbb{N}} \mathtt{suc}\ n) \simeq \bot$$

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By requiring **unifiers** to be **equivalences**:

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Next, we'll explore how this idea can help us.

Any questions so far?

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Time for the interesting bits!

- Equations between types
- Heterogeneous equations
- Equations on indexed datatypes
- Equations between equations

Equations between types

Types are first-class terms of type $\texttt{Type}$:
$\texttt{Bool} : \texttt{Type}$, $\mathbb{N} : \texttt{Type}$, $\mathbb{N} \to \mathbb{N} : \texttt{Type}$, $\ldots$

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We can form equations between types,
e.g. $\texttt{Bool} \equiv_{\texttt{Type}} \texttt{Bool}$.

Q: Can we apply the `deletion` rule?

A: Depends on which type theory we use!

The univalence axiom (2009)

Vladimir
Voevodsky

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“Isomorphic types can be identified.”

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"Isomorphic types can be identified."

$$(A \equiv B) \simeq (A \simeq B)$$

The univalence axiom (2009)

`Bool` is equal to `Bool` in two ways:

`Bool`

`true`	`false`

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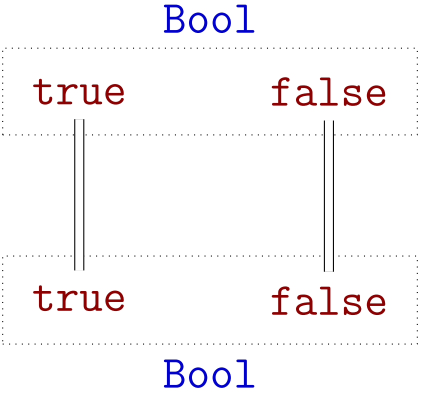

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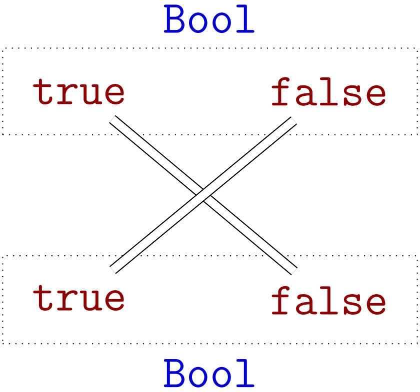

Limiting the deletion rule

The `deletion` rule does not always hold: there might be multiple proofs of $x \equiv_A x$.

E.g. `Bool` $\equiv_{\texttt{Type}}$ `Bool` has two elements.

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E.g. `Bool` $\equiv_{\texttt{Type}}$ `Bool` has two elements.

We cannot use `deletion` in this case!

Heterogeneous equations

$\Sigma_{n:\mathbb{N}}\texttt{Vec}\ A\ n$ is the type of pairs (n, xs) where $n : \mathbb{N}$ and $xs : \texttt{Vec}\ A\ n$.

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$\Sigma_{n:\mathbb{N}}\texttt{Vec}\ A\ n$ is the type of pairs (n, xs) where $n : \mathbb{N}$ and $xs : \texttt{Vec}\ A\ n$.

$$(e : (0, \texttt{nil}) \equiv_{\Sigma_{n:\mathbb{N}}\texttt{Vec}\ A\ n} (1, \texttt{cons}\ 0\ x\ xs))$$

$$\wr|$$

$$(e_1 : 0 \equiv_{\mathbb{N}} 1)(e_2 : \texttt{nil} \equiv_{\texttt{Vec}\ A\ ???} \texttt{cons}\ 0\ x\ xs)$$

Heterogeneous equations

$\Sigma_{n:\mathbb{N}}\texttt{Vec}\ A\ n$ is the type of pairs (n, xs) where $n : \mathbb{N}$ and $xs : \texttt{Vec}\ A\ n$.

$$(e : (0, \texttt{nil}) \equiv_{\Sigma_{n:\mathbb{N}}\texttt{Vec}\ A\ n} (1, \texttt{cons}\ 0\ x\ xs))$$

$$\wr|$$

$$(e_1 : 0 \equiv_{\mathbb{N}} 1)(e_2 : \texttt{nil} \equiv_{\texttt{Vec}\ A\ ???} \texttt{cons}\ 0\ x\ xs)$$

What is the type of e_2?

Heterogeneous equations

Solution: use equation variables as placeholders for their solutions:

$$(e : (0, \mathtt{nil}) \equiv_{\Sigma_{n:\mathbb{N}} \mathtt{Vec}\ A\ n} (1, \mathtt{cons}\ 0\ x\ xs))$$

$$\wr|$$

$$(e_1 : 0 \equiv_{\mathbb{N}} 1)(e_2 : \mathtt{nil} \equiv_{\mathtt{Vec}\ A\ e_1} \mathtt{cons}\ 0\ x\ xs)$$

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$$(e_1 : 0 \equiv_{\mathbb{N}} 1)(e_2 : \texttt{nil} \equiv_{\texttt{Vec}\ A\ e_1} \texttt{cons}\ 0\ x\ xs)$$

This is called a *telescopic equality*.

Be careful with heterogeneous equations!

$$(e : (\texttt{Bool}, \texttt{true}) \equiv_{\Sigma_{A:\texttt{Type}} A} (\texttt{Bool}, \texttt{false}))$$

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$$\wr|$$

$$\bot$$

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$$\not\wr\!|$$

$$\bot$$

The `conflict` rule does not apply!

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$$\wr|$$

$$\bot$$

Whether a unification rule can be applied depends on the **type** of the equation!

Injectivity for indexed data

Do standard unification rules apply to constructors of indexed datatypes?

$$(e : \texttt{cons}\ n\ x\ xs \equiv_{\texttt{Vec}\ A\ (\texttt{suc}\ n)} \texttt{cons}\ n\ y\ ys)$$

$$\wr|$$

???

Injectivity for indexed data

Idea: simplify equations between indices together with equation between constructors:

$$(e_1 : \texttt{suc}\ k \equiv_{\mathbb{N}} \texttt{suc}\ n)$$
$$(e_2 : \texttt{cons}\ k\ x\ xs \equiv_{\texttt{Vec}\ A\ e_1} \texttt{cons}\ n\ y\ ys)$$

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$$\wr|$$
$$(e_1' : k \equiv_{\mathbb{N}} n)(e_2' : x \equiv_A y)$$
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Injectivity for indexed data

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$$\cong$$
$$(e_1' : k \equiv_{\mathbb{N}} n)(e_2' : x \equiv_A y)$$
$$(e_3' : xs \equiv_{\mathtt{Vec}\ A\ e_1} ys)$$

Length of the `Vec` must be *fully general*: must be an equation variable.

The image datatype

The type $\texttt{Im}\ f\ y$ consists of elements $\texttt{image}\ x$ such that $f\ x = y$:

$$\textbf{data}\ \texttt{Im}\ (f : A \to B) : B \to \texttt{Type}\ \textbf{where}$$
$$\quad \texttt{image} : (x : A) \to \texttt{Im}\ f\ (f\ x)$$

Solving unsolvable equations

$$(x_1\ x_2 : A)(e_1 : f\ x_1 \equiv_B f\ x_2)$$
$$(e_2 : \texttt{image}\ x_1 \equiv_{\texttt{Im}\ f\ e_1} \texttt{image}\ x_2)$$

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Solving unsolvable equations

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$$\wr|$$
$$(x_1\ x_2 : A)(e : x_1 \equiv_A x_2)$$
$$\wr|$$
$$(x_1 : A)$$

What if the indices are not fully general?

$(e : \texttt{cons}\ n\ x\ xs \equiv_{\texttt{Vec}\ A\ (\texttt{suc}\ n)} \texttt{cons}\ n\ y\ ys)$

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$$(e_2 : \texttt{cons}\ n\ x\ xs \equiv_{\texttt{Vec}\ A\ e_1} \texttt{cons}\ n\ y\ ys)$$

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$$\wr|$$

$$(e'_1 : n \equiv_{\mathbb{N}} n)(e'_2 : x \equiv_A y)(e'_3 : xs \equiv_{\texttt{Vec}\ A\ e'_1} ys)$$

$$(p : \texttt{cong}\ \texttt{suc}\ e'_1 \equiv_{\texttt{suc}\ n \equiv_{\mathbb{N}} \texttt{suc}\ n} \texttt{refl})$$

What if the indices are not fully general?

$$(e : \texttt{cons}\ n\ x\ xs \equiv_{\texttt{Vec}\ A\ (\texttt{suc}\ n)} \texttt{cons}\ n\ y\ ys)$$

$$\wr|$$

$$(e_1 : \texttt{suc}\ n \equiv_{\mathbb{N}} \texttt{suc}\ n)$$

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$$(p : e_1 \equiv_{\texttt{suc}\ n \equiv_{\mathbb{N}} \texttt{suc}\ n} \texttt{refl})$$

$$\wr|$$

$$(e_1' : n \equiv_{\mathbb{N}} n)(e_2' : x \equiv_A y)(e_3' : xs \equiv_{\texttt{Vec}\ A\ e_1'} ys)$$

$$(p : \texttt{cong}\ \texttt{suc}\ e_1' \equiv_{\texttt{suc}\ n \equiv_{\mathbb{N}} \texttt{suc}\ n} \texttt{refl})$$

Higher-dimensional equations

$$(e_1' : n \equiv_{\mathbb{N}} n)(e_2' : x \equiv_A y)(e_3' : xs \equiv_{\texttt{Vec}\ A\ e_1'} ys)$$
$$(p : \texttt{cong}\ \texttt{suc}\ e_1' \equiv_{\texttt{suc}\ n \equiv_{\mathbb{N}} \texttt{suc}\ n} \texttt{refl})$$

We call an equation between equality proofs (e.g. p) a **higher-dimensional equation**.

How to solve higher-dimensional equations?

Existing unification rules do not apply…

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Existing unification rules do not apply…

We solve the problem in three steps:

1. lower the dimension of equations
2. solve lower-dimensional equations
3. lift unifier to higher dimension

Step 1: lower the dimension of equations

We replace all equation variables by regular variables: instead of

$$(e_1 : n \equiv_{\mathbb{N}} n)(e_2 : x \equiv_A y)(e_3 : xs \equiv_{\texttt{Vec}\ A\ e_1} ys)$$
$$(p : \texttt{cong}\ \texttt{suc}\ e_1 \equiv_{\texttt{suc}\ n \equiv_{\mathbb{N}} \texttt{suc}\ n} \texttt{refl})$$

let's first consider

$$(k : \mathbb{N})(u : A)(us : \texttt{Vec}\ A\ k)$$
$$(e : \texttt{suc}\ k \equiv_{\mathbb{N}} \texttt{suc}\ n)$$

Step 2: solve lower-dimensional equations

This gives us an equivalence f of type

$$(k : \mathbb{N})(u : A)(us : \texttt{Vec}\ A\ k)$$
$$(e : \texttt{suc}\ k \equiv_{\mathbb{N}} \texttt{suc}\ n)$$
$$\wr|$$
$$(u : A)(us : \texttt{Vec}\ A\ n)$$

Step 3: lift unifier to higher dimension

We lift f to an equivalence $f^{\uparrow}$ of type

$$(e_1 : n \equiv_{\mathbb{N}} n)(e_2 : x \equiv_A y)$$
$$(e_3 : xs \equiv_{\texttt{Vec}\ A\ e_1} ys)$$
$$(p : \texttt{cong suc}\ e_1 \equiv_{\texttt{suc}\ n \equiv_{\mathbb{N}} \texttt{suc}\ n} \texttt{refl})$$
$$\wr|$$
$$(e_2 : x \equiv_A y)(e_3 : xs \equiv_{\texttt{Vec}\ A\ n} ys)$$

Final result of steps 1-3

$$(e : \texttt{cons}\ n\ x\ xs \equiv_{\texttt{Vec}\ A\ (\texttt{suc}\ n)} \texttt{cons}\ n\ y\ ys)$$

$$\wr|$$

$$(e_2 : x \equiv_A y)(e_3 : xs \equiv_{\texttt{Vec}\ A\ n} ys)$$

Final result of steps 1-3

$$(e : \texttt{cons}\ n\ x\ xs \equiv_{\texttt{Vec}\ A\ (\texttt{suc}\ n)} \texttt{cons}\ n\ y\ ys)$$

$$\wr|$$

$$(e_2 : x \equiv_A y)(e_3 : xs \equiv_{\texttt{Vec}\ A\ n}\ ys)$$

This is the **forcing rule** for `cons`.

Lifting equivalences: (mostly) general case

Theorem. If we have an equivalence f of type

$$(x : A)(e : b_1\ x \equiv_{B\ x} b_2\ x) \simeq C$$

we can construct $f^{\uparrow}$ of type

$$(e : u \equiv_A v)(p : \texttt{cong}\ b_1\ e \equiv_{r \equiv_{B\ e} s} \texttt{cong}\ b_2\ e)$$
$$\wr|$$
$$(e' : f\ u\ r \equiv_C f\ v\ s)$$

Implementation in Agda

This is all used by Agda to check definitions by dependent pattern matching.

- More general than before
- Fixed many bugs
- Implementation matches theory

You can try it for yourself:
`wiki.portal.chalmers.se/agda`

Conclusion

Unification rules should return **evidence** of their correctness.

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A most general unifier can be represented internally as an **equivalence**.

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Unification rules should return **evidence** of their correctness.

A most general unifier can be represented internally as an **equivalence**.

Unification cannot ignore the **types**!

Questions?

If you want to know more, you can:

- Try out Agda:
  `wiki.portal.chalmers.se/agda`
- Look at the source:
  `github.com/agda/agda`
- Read my thesis:
  Dependent pattern matching and proof-relevant unification (2017)

Two applications of unification

Filling in implicit arguments

Checking definitions by pattern matching

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Filling in implicit arguments

- Higher order

Checking definitions by pattern matching

- First order

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- ‘Syntactic’

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Focus of this talk

Two notions of equality

Definitional equality

$$x = y : A$$

- Weaker

Propositional equality

$$e : x \equiv_A y$$

- Stronger

Two notions of equality

Definitional equality

$$x = y : A$$

- Weaker
- Decidable

Propositional equality

$$e : x \equiv_A y$$

- Stronger
- Undecidable

Two notions of equality

Definitional equality	Propositional equality
$x = y : A$	$e : x \equiv_A y$

Definitional equality:

- Weaker
- Decidable
- Meta-theoretic

Propositional equality:

- Stronger
- Undecidable
- Internal to theory

Two notions of equality

Definitional equality

$$x = y : A$$

- Weaker
- Decidable
- Meta-theoretic
- Implicit

Propositional equality

$$e : x \equiv_A y$$

- Stronger
- Undecidable
- Internal to theory
- Explicit